CONCOMITANTS OF GENERALIZED ORDER STATISTICS FOR PSEUDO RAYLEIGH DISTRIBUTION

Saman Hanif

Department of Statistics, Faculty of Sciences, King Abdul Aziz University, Jeddah, Saudi Arabia

saman.hanif@hotmail.com

In this paper we have studied the distributional properties of Concomitants of Generalized Order Statistics for Pseudo Rayleigh Distribution. Expressions for Moments of Concomitants have also been obtained.

Key Words: *Concomitants, Generalized Order Statistics, Pseudo Distributions*

1. INTRODUCTION

Filus and Filus (2006) introduced a new class of multivariate distributions as joint distribution of linear combinations of independent random variables and named the resulting class as pseudo distributions. Filus and Filus (2006) have studied pseudo Weibull and pseudo Gamma distributions. Hanif (2007) has presented the pseudo distributions as compound distribution of two random variables and has studied pseudo Gaussian and pseudo Weibull distribution. Shahbaz and Shahbaz (2009) have defined the bivariate pseudo Rayleigh distribution as compound distribution of two Rayleigh variates. The joint density function of pseudo Rayleigh distribution given by Shahbaz and Shahbaz (2009) is

$$f\left(x,y;\alpha\right)=4\alpha x^{3}y\exp\left\{-x^{2}\left(\alpha+y^{2}\right)\right\};\quad \alpha,x,y>0. \tag{1.1}$$

Shahbaz and Shahbaz (2009) have studied the concomitants of order statistics; given by Ahsanullah and Nevzorov (2001); for distribution (1.1).

Kamps (1995) has defined a unified model of ordered random variables and named the model as Generalized Order Statistics (GOS). The density function of rth GOS is given by Kamps (1995) as

$$f_{r:n,m,k}\left(x\right)=\frac{C_{r-1}}{\left(r-1\right)!}f\left(x\right)\left\{1-F\left(x\right)\right\}^{\gamma_r-1}\times g_m^{r-1}\left[F\left(x\right)\right], \tag{1.2}$$

where $C_{r-1}=\prod_{j=1}^{r}\gamma_j\,;r=1,2,\ldots,n$,

$$g_m\left(x\right)=h_m\left(x\right)-h_m\left(0\right)$$

and

$$=\begin{cases}\left[1-\left(1-x\right)^{m+1}\right]\Big/\left(m+1\right); & m\neq-1\\ -\ln\left(1-x\right) & m=-1.\end{cases}$$

We also have

$$h_m\left(x\right)=\begin{cases}-\left(1-x\right)^{m+1}\Big/\left(m+1\right); & m\neq-1\\ -\ln\left(1-x\right) & m=-1.\end{cases}$$

Kamps (1995) has further shown that the joint density function of two GOS $X_{r:n,m,k}$ and $X_{s:n,m,k}$ for $r<s$ is given as

$$\begin{aligned}f_{r,s:n,m,k}\left(x_1,x_2\right)=&\frac{C_{s-1}}{\left(r-1\right)!\left(s-r-1\right)!}f\left(x_1\right)f\left(x_2\right)\\ &\times\left\{1-F\left(x_1\right)\right\}^{m}g_m^{r-1}\left\{F\left(x_1\right)\right\}\left\{1-F\left(x_2\right)\right\}^{\gamma_s-1}\\ &\times\left[h_m\left\{F\left(x_2\right)\right\}-h_m\left\{F\left(x_1\right)\right\}\right]^{s-r-1}\\ &;-\infty<x_1<x_2<\infty.\end{aligned} \tag{1.3}$$

The model of GOS defined by Kamps (1995) provide several models of ordered data as a special case. Order Statistics appear as special case of GOS for $m=0$ and $k=1$. The Record Statistics defined by Chandler (1952) appear as a special case of GOS for for $m=-1$.

Concomitants of ordered random variables appear naturally when sample is available from a bivariate distribution and data is ordered with respect to one variable. Ahsanullah and Nevzorov (2001) have discussed concomitants of GOS. The density function of rth concomitant of GOS is given by Ahsanullah and Nevzorov (2001) as

$$f_{[r:n,m,k]}\left(y\right)=\int_{-\infty}^{\infty}f\left(y\,|\,x\right)f_{r:n,m,k}\left(x\right)dx, \tag{1.4}$$

where $f_{r:n,m,k}\left(x\right)$ is given in (1.2). The joint distribution of two concomitants of GOS is given by Ahsanullah and Nevzorov (2001) as

$$f_{[r,s:n,m,k]}\left(y_1,y_2\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f\left(y_1\,|\,x_1\right)f\left(y_2\,|\,x_2\right)\times f_{r,s:n,m,k}\left(x_1,x_2\right)dx_1dx_2, \tag{1.5}$$

where $f_{r,s:n,m,k}\left(x_1,x_2\right)$ is given in (1.3).

Several authors have studied the concomitants of GOS. Ahsanullah and Beg (2006) have studied concomitants of GOS for Gumbel Bivariate Exponential distribution. Concomitants of GOS for Gumbel bivariate family of distributions has been studied by Beg and Ahsanullah (2008). Nayabuddin (2013) has studied concomitants of GOS for bivariate Lomax distribution. Hanif and Shahbaz (2016) has studied the concomitants of GOS for a bivariate exponential distribution.

In this paper we have studied the concomitants of GOS for pseudo Rayleigh distribution given in (1.1) by Shahbaz and Shahbaz (2009).

2. DISTRIBUTION OF *r*th CONCOMITANTS OF GOS

In this section we have obtained the distribution of rth concomitants of GOS for Rayleigh distribution. For this consider we consider the joint distribution given in (1.1). In order to obtain the distribution of concomitants we first obtain the marginal distribution of X and conditional distribution of Y given X from (1.1). The marginal distribution of X is readily obtained from (1.1) as

$$f\left(x;\alpha\right)=2\alpha x\exp\left(-\alpha x^2\right), \qquad (2.1)$$

and the conditional distribution of Y given X is obtained from (1.1) and (2.1) as

$$f\left(y\mid x\right)=2x^2y\exp\left(-x^2y^2\right). \qquad (2.2)$$

We now obtain the distribution of rth GOS for the distribution (2.1). For this we first note that

$$F\left(x\right)=1-\exp\left(-\alpha x^2\right)\Rightarrow 1-F\left(x\right)=\exp\left(-\alpha x^2\right).$$

So $$g_m\left[F\left(x\right)\right]=\frac{1}{m+1}\left[1-\exp\left\{-\alpha\left(m+1\right)x^2\right\}\right]$$

and

$$g_m^{r-1}\left[F\left(x\right)\right]=\frac{1}{\left(m+1\right)^{r-1}}\left[1-\exp\left\{-\alpha\left(m+1\right)x^2\right\}\right]^{r-1}$$
$$=\frac{1}{\left(m+1\right)^{r-1}}\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\times\exp\left\{-\alpha\left(m+1\right)ix^2\right\}.$$

The distribution of rth GOS for (2.1) is therefore:

$$f_{r:n,m,k}\left(x\right)=\frac{C_{r-1}}{\left(r-1\right)!}2\alpha x\exp\left(-\alpha x^2\right)\times\exp\left\{-\alpha x^2\left(\gamma_r-1\right)\right\}\frac{1}{\left(m+1\right)^{r-1}}\times\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\exp\left\{-\alpha\left(m+1\right)ix^2\right\}$$

or

$$f_{r:n,m,k}\left(x\right)=\frac{C_{r-1}}{\left(r-1\right)!\left(m+1\right)^{r-1}}\sum_{i=0}^{r-1}\left(-1\right)^i\times\binom{r-1}{i}2\alpha x\exp\left(-\alpha w_1x^2\right); \qquad (2.3)$$

$$w_1=\left\{\left(m+1\right)i+\gamma_r\right\}$$

Now using (2.2) and (2.3) in (1.4), the distribution of rth concomitant of GOS for (2.1) is:

$$f_{[r:n,m,k]}\left(y\right)=\int_0^\infty 2x^2y^2\exp\left(-x^2y^2\right)\frac{C_{r-1}}{\left(r-1\right)!\left(m+1\right)^{r-1}}\times\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}2\alpha x\exp\left(-\alpha w_1x^2\right)dx$$

which after simplifications becomes

$$f_{[r:n,m,k]}\left(y\right)=\frac{C_{r-1}}{\left(r-1\right)!\left(m+1\right)^{r-1}}\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\times\frac{2\alpha y}{\left(y^2+\alpha w_1\right)^2};\ y>0 \qquad (2.4)$$

The pth moment of rth concomitant of GOS is obtained below.

Now

$$\mu_{[r:n,m,k]}^p=\frac{C_{r-1}}{\left(r-1\right)!\left(m+1\right)^{r-1}}\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\times\int_0^\infty\frac{2\alpha y}{\left(y^2+\alpha w_1\right)^2}dy$$

which after simplifications become

$$\mu_{[r:n,m,k]}^p=\frac{C_{r-1}}{\left(r-1\right)!\left(m+1\right)^{r-1}}\frac{\alpha p}{2}\Gamma\left(p/2\right)\Gamma\left(p/2-1\right)\times\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\left(\alpha w_1\right)^{p/2-1}. \qquad (2.5)$$

The moments can be computed for specific values of the parameters.

3. JOINT DISTRIBUTION OF *r*th and *s*th CONCOMITANTS OF GOS

In this section we have obtained the joint distribution of two concomitants of GOS for the bivariate Rayleigh distribution. For this we first obtain the joint distribution of two GOS for the distribution (2.1). Now using (1.3), the joint distribution of two GOS for Rayleigh distribution is

$$f_{r,s:n,m,k}\left(x_1,x_2\right)=\frac{C_{s-1}}{\left(r-1\right)!\left(s-r-1\right)!}4\alpha^2x_1x_2\exp\left(-\alpha x_1^2\right)\times\exp\left(-\alpha x_2^2\right)\exp\left(-\alpha mx_1^2\right)\frac{1}{\left(m+1\right)^{r-1}}\times\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\exp\left\{-\alpha\left(m+1\right)ix_1^2\right\}\times\exp\left\{-\alpha x_2^2\left(\gamma_s-1\right)\right\}\frac{1}{\left(m+1\right)^{s-r-1}}\times\left[\exp\left\{-\alpha\left(m+1\right)x_1^2\right\}-\exp\left\{-\alpha\left(m+1\right)x_2^2\right\}\right]^{s-r-1};$$

which after simplifications become

$$f_{r,s:n,m,k}\left(x_1,x_2\right)=\frac{C_{s-1}}{\left(r-1\right)!\left(s-r-1\right)!\left(m+1\right)^{s-2}}\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\left(-1\right)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\times 4\alpha^2x_1x_2\exp\left(-\alpha w_2x_1^2\right)\exp\left(-\alpha w_3x_2^2\right); \qquad (3.1)$$
$$0<x_1<x_2<\infty,$$

where

$$w_2=\left\{\left(m+1\right)\left(s-r-j+i\right)\right\};\ w_3=\left\{\left(m+1\right)j+\gamma_s\right\}.$$

Now using (3.1) in (1.5) the joint distribution of two concomitants of GOS is obtained below.

$$f_{[r,s:n,m,k]}\left(y_1,y_2\right)=4\int_0^\infty\int_{x_1}^\infty x_1^2y_1\exp\left(-x_1^2y_1^2\right)x_2^2y_2\times\exp\left(-x_2^2y_2^2\right)\frac{C_{s-1}}{\left(r-1\right)!\left(s-r-1\right)!\left(m+1\right)^{s-2}}\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\left(-1\right)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}4\alpha^2x_1x_2\times\exp\left(-\alpha w_2x_1^2\right)\exp\left(-\alpha w_3x_2^2\right)dx_2dx_1$$

or

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}$$
$$\times\binom{r-1}{i}\binom{s-r-1}{j}\alpha^2\beta_1\beta_2^2y_1^{\beta_2-1}y_2^{\beta_2-1}$$
$$\times\int_0^\infty x_1^{2\beta_1-1}\exp\left\{-x_1^{\beta_1}\left(y_1^{\beta_2}+\alpha w_2\right)\right\}I(x_2)\,dx_1$$

where $I(x_2)=2\int_{x_1}^\infty x_2^3\exp\left\{-x_2^2\left(y_2^2+\alpha w_3\right)\right\}dx_2$.

Now making the transformation $w=x_2^2\left(y_2^2+\alpha w_3\right)$ we have

$$I(x_2)=\frac{1}{\left(y_2^2+\alpha w_3\right)^2}\left\{1+\left(y_2^2+\alpha w_3\right)x_1^2\right\}$$
$$\times\exp\left\{-x_1^2\left(y_2^2+\alpha w_3\right)\right\}.$$

We therefore have

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times 8\alpha^2y_1y_2\frac{1}{\left(y_2^2+\alpha w_3\right)^2}\int_0^\infty x_1^3\exp\left\{-x_1^2\left(y_1^2+\alpha w_2\right)\right\}$$
$$\times\left\{1+\left(y_2^2+\alpha w_3\right)x_1^2\right\}\exp\left\{-x_1^2\left(y_2^2+\alpha w_3\right)\right\}dx_1,$$

which after simplifications become

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times 4\alpha^2\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \quad (3.2)$$
$$\times\frac{y_1y_2\left\{y_1^2+3y_2^2+\alpha\left(w_2+w_3\right)\right\}}{\left(y_2^2+\alpha w_3\right)^2\left\{y_1^2+y_2^2+\alpha\left(w_2+w_3\right)\right\}^3}.$$

The product moments can be obtained by using (3.2).

REFERENCES:

1. Ahsanullah, M. and Beg, M. I. (2007) Concomitant of generalized order statistics in Gumbel bivariate Exponential distribution, *J. Stat. Th. and App.*, Vol. 6, 118–132.
2. Ahsanullah, M. and Nevzorov, V. B. (2001) Ordered Random Variables, Nova Science Publishers, USA.
3. Beg, M.I. and Ahsanullah, M. (2008). Concomitants of generalized order statistics from Farlie-Gumbel-Morgensterm distributions. *Statistical Methodology*, 5, 1–20.
4. Chandler, K. N. (1952). The distribution and frequency of record values, *J. Royal Statist. Soc. B*, 14, 220 – 228.
5. Filus, J.K. and Filus, L.Z. (2006). On some new classes of Multivariate Probability Distributions. *Pak. J. Statist*. 22(1), 21–42.
6. Hanif. S. (2007) *Concomitants of Ordered Random Variables*, Unpublished PhD Thesis, NCBA&E.
7. Hanif, S. and Shahbaz, M. Q. (2016) Concomitants of Generalized Order Statistics for a Bivariate Exponential Distribution, *Pak. J. Stat. & OR.*, Vol. 12(2), 227–234.
8. Kamps, U. (1995). A concept of generalized order statistics, *J. Statist. Plann. Inference*, 48, 1-23.
9. Nayabuddin (2013) Concomitants of generalized order statistics from bivariate Lomax distribution, *ProbStat Forum*, Vol. 6, 73–88.
10. Shahbaz, M. Q. and Shahbaz, S. (2009) Order Statistics and concomitants of Bivariate Pseudo Rayleigh distribution, *World App. Sci. J.*, Vol 7(7), 826–828.